A Non-Proportional Premium Rating Method for Construction Risks

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(Revised)

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Correct pricing of non-proportional insurance for construction risks must consider not only how property values build up over time but also how the Probable Maximum Loss (PML) changes. A proper method is developed with analysis of specific cases of interest.

DEFINITIONS

T	Period of Insurance
$v(t)$	Total value exposed at time t, $0 \leq t \leq T$
$m(t)$	Probable Maximum Loss (PML) at time t, $0 \leq t \leq T$
V	$\max v(t) = v(T)$ .... i.e., *Total Insured Value*
M	$\max m(t)$ .... i.e., *Maximum PML* or simply *PML*
V_0	Bottom of the layer[1] .... i.e., *attachment point*
V_1	Top of the layer .... i.e., $V_1 - V_0$ is the *layer stretch*
r	Premium per unit value per unit time (or *premium rate* per unit time)
P	Total premium for the risk
$E(x)$	Exposure rating curve applicable to the exposure[2]

1. THE GENERAL CASE

Note that $rv(t)dt$ is the premium for the risk between t and $t + dt$; $rv(t)$ is the ***premium density***.[3] The total premium P for the risk is:

$$P = \int_0^T rv(t)dt$$

Hence we can write:

$$r = P\left\{\int_0^T v(t)dt\right\}^{-1} \tag{1}$$

Here we implicitly assumed that r is a constant. In fact, r may vary with time if the perils, coverages or nature of the property insured change. For example, if:

- Construction methods and materials change significantly during the project term;
- The project transitions from a construction phase to a testing phase;
- Windstorm exposure changes throughout the year.

[1] We ignore underlying deductibles. If a deductible D applies, V_0 and V_1 can be replaced with $V_0 + D$ and $V_1 + D$, but then P must be replaced with the premium P' that would be charged with a zero deductible. This may not be easy to determine. It is not correct to write $P = \{1 - E(D/M)\}P'$: the curves $E(x)$ are not suitable for thin layers close to zero, as they do not account for loss frequency and reinstatement.

[2] For $0 \leq x \leq 1$, $E(x)$ is the proportion of premium for an exposure with PML Q that should be allocated to the primary layer xQ. See Guggisberg, D. (2004) *Exposure Rating.* Swiss Re.

[3] If $P(t)$ is the premium for the interval $[0, t]$, then $P'(t)$ is the premium density, since the premium for the interval $\sigma \leq t \leq \tau$ is given by $P(\tau) - P(\sigma) = \int_\sigma^\tau P'(t)\,dt$. Therefore, we may *define* $r \equiv P'(t)/v(t)$.

We will deal with various perils and coverages separately, so we may assume that ***r* is constant** unless otherwise stated. So if $\mathcal{E}_i$ is the exposure arising from a particular peril (fire, earthquake, flood, etc.) and/or during a particular phase of the project (foundations, erection, testing, etc.) between $t = t_a$ and $t = t_b$, for which a premium P_i is paid, then

$$r_i = P_i \left\{ \int_{t_a}^{t_b} v(t) dt \right\}^{-1}$$

In construction insurance, the ***Probable Maximum Loss*** (PML) is a loss estimate based on a "plausible worst case" scenario (equivalent to the *Maximum Foreseeable Loss* or *Estimated Maximum Loss* used elsewhere in insurance). A commonly accepted definition is:[4]

> *Probable Maximum Loss is an estimate of the maximum loss which could be sustained by the insurers as a result of any one occurrence considered by the underwriter to be within the realms of probability. This ignores such coincidences and catastrophes which are remote possibilities, but which remain highly improbable.*

Throughout this paper we assume that any exposure in excess of the PML attracts no premium. Therefore, if there is an exposure with PML m during an interval Δt, then a primary layer with limit $Q \leq m$ deserves a proportion $E(Q/m)$ of the premium for the interval Δt, where $E(x)$ is a suitably chosen exposure rating curve. We will apply this principle repeatedly.

We may assume that $V_1 \leq M$ (otherwise we can redefine $V_1 = M$). We also assume for the moment that the PML function $m(t)$ is non-decreasing (which is usually the case), as shown in Figure 1 below. (This assumption will be relaxed later.)

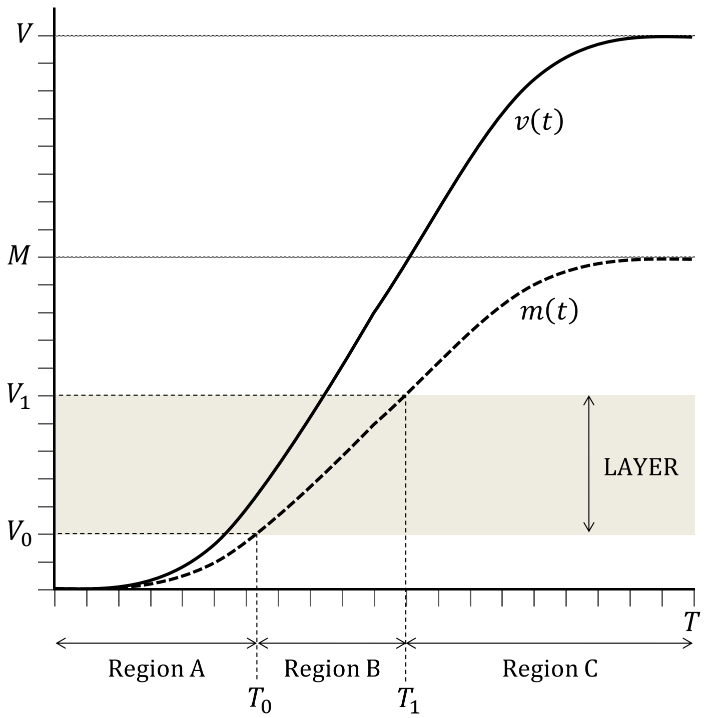


FIGURE 1

[4] See Heller, H. et al. (2002) *Working Group Paper for the Possible Maximum Loss Assessment of Civil Engineering Projects*. IMIA Paper WGP 19(02)E.

Let $T_0 = \inf\{t : m(t) > V_0\}$ be the time when $m(t)$ enters the layer, and $T_1 = \sup\{t : m(t) \leq V_1\}$ be the time when $m(t)$ exits the layer. If m is continuous, T_0 and T_1 can be calculated by solving the equations:

$$\begin{aligned} m(T_0) &= V_0 \\ m(T_1) &= V_1 \end{aligned} \tag{2}$$

Let L_{A}, L_{B} and L_{C} be the layer premiums for the regions A, B and C. Obviously, $L_{\mathrm{A}} = 0$ since $m(t) \leq V_0$. Looking at region B, we see that:

$$L_{\mathrm{B}} = \int_{T_0}^{T_1} \left\{1 - E\left(\frac{V_0}{m(t)}\right)\right\} rv(t)dt$$

The first term $1 - E(V_0/m(t))$ is the proportion of premium that should be allocated to the layer from V_0 to $m(t)$, and the second term $rv(t)dt$ is the premium for the interval $[t, t + dt]$.

Similarly for region C:

$$L_{\mathrm{C}} = \int_{T_1}^{T} \left\{E\left(\frac{V_1}{m(t)}\right) - E\left(\frac{V_0}{m(t)}\right)\right\} rv(t)dt$$

The first term $E(V_1/m(t)) - E(V_0/m(t))$ is the proportion of premium that should be allocated to the layer from V_0 to V_1, and the second term $rv(t)dt$ is again the premium for the interval $[t, t + dt]$.

Combining terms, the layer premium $L = L_{\mathrm{A}} + L_{\mathrm{B}} + L_{\mathrm{C}}$ is:

$$L = \int_{T_0}^{T_1} \left\{1 - E\left(\frac{V_0}{m(t)}\right)\right\} rv(t)dt + \int_{T_1}^{T} \left\{E\left(\frac{V_1}{m(t)}\right) - E\left(\frac{V_0}{m(t)}\right)\right\} rv(t)dt \tag{3}$$

or, rearranging terms:

$$L = \int_{T_0}^{T_1} rv(t)dt - \int_{T_0}^{T} E\left(\frac{V_0}{m(t)}\right) rv(t)dt + \int_{T_1}^{T} E\left(\frac{V_1}{m(t)}\right) rv(t)dt \tag{4}$$

Equation (3) can be written more compactly and generally as follows:

$$\boxed{L = \int_{0}^{T} \left\{E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right)\right\} rv(t)dt \tag{5}}$$

since

$$E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right) = \begin{cases} 0 & m(t) < V_0 \\ 1 - E(V_0/m(t)) & V_0 \leq m(t) < V_1 \\ E(V_1/m(t)) - E(V_0/m(t)) & V_1 \leq m(t) \end{cases}$$

This form is suitable and preferable for programming a computer, since L can be calculated without solving (2) for T_0 and T_1. Note that (5) ***is valid for any PML function*** $\boldsymbol{m(t)}$, regardless of whether it is non-decreasing or continuous.

We will see in Section 4 that when $v(t)$ is a straight line or an S-shaped curve (or any function that is symmetrical under a 180° rotation[5]), then:

$$r = \frac{2P}{VT}$$

In this case, (5) simplifies to:

$$\boxed{L = \frac{2P}{VT}\int_0^T \left\{E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right)\right\} v(t)dt \qquad (6)}$$

Primary layers. Note that (5) or (6) may also be used to price a primary layer by setting $V_0 = 0$, causing the second term in curly brackets to vanish. From (5):

$$L_{\text{primary } V_1} = \int_0^T E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) r v(t)dt \qquad (7)$$

and when $v(t)$ is symmetrical under a 180° rotation:

$$L_{\text{primary } V_1} = \frac{2P}{VT}\int_0^T E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) v(t)dt \qquad (8)$$

In general, $v(t)$ is given by some variety of S-shaped curve, $m(t)$ is determined by the underwriter, and $E(x)$ is an increasing, concave function of the form:[6]

$$E_c(x) = \frac{\ln\left[\frac{(\alpha-1)\beta + (1-\alpha\beta)\beta^x}{1-\beta}\right]}{\ln \alpha\beta} \qquad (9)$$

where

$$\alpha = \alpha(c) = e^{(0.78-0.12c)c}$$

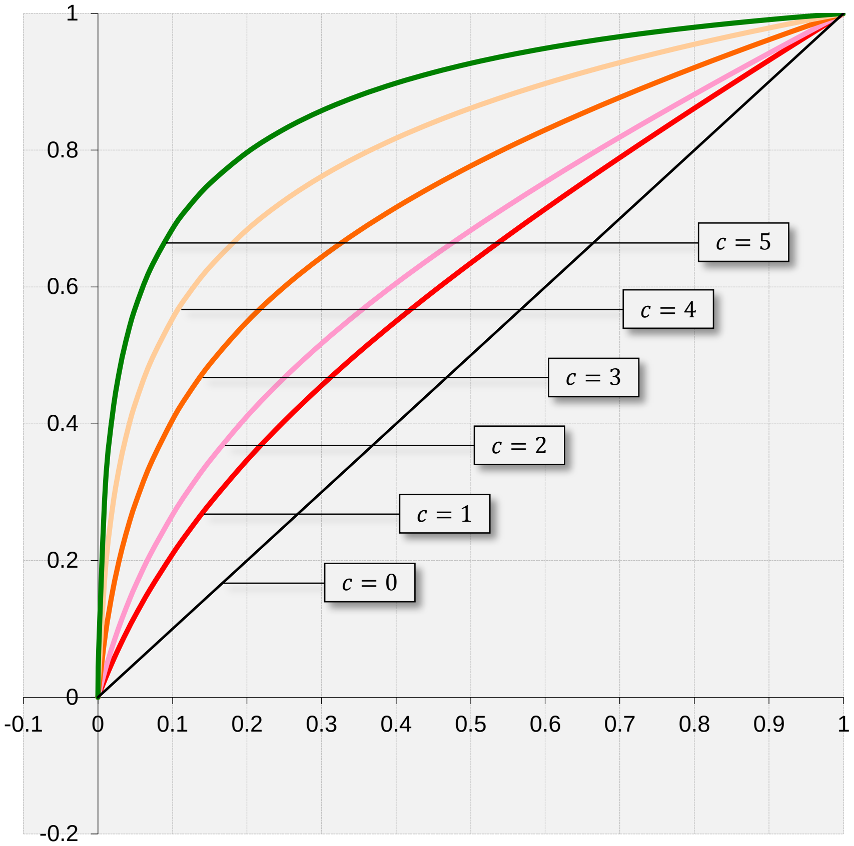
$$\beta = \beta(c) = e^{3.1-0.15(1+c)c}$$

and $c \geq 0$ is a free parameter.

[5] Symmetry under a 180° rotation means $v(t) + v(T-t) = V$.

[6] See Bernegger, S. (1997) The Swiss Re Exposure Curves and the MBBEFD Distribution Class. *ASTIN Bulletin,* 27(1), pp. 99-111.

The chart below shows $E_c(x)$ for integer c, $0 \le c \le 5$. $E_c(x)$ for $c = 1, 2, 3, 4$ are known as the ***Swiss Re Y_c curves***. $E_5(x)$ is known as the ***Lloyd's curve***.



2. Linear build-up of value

When v is linear, that is $v(t) = (V/T)t$, we have from (1):

$$r = \frac{2P}{VT} \tag{10}$$

The general formula (5) then becomes:

$$L = \int_0^T \left\{E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right)\right\} \cdot \frac{2P}{VT} \cdot \frac{Vt}{T} dt$$

So:

$$\frac{L}{P} = \frac{2}{T^2} \int_0^T t\left\{E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right)\right\} dt \tag{11}$$

Alternatively, using the explicit formulation in (4), we have:

$$\frac{L}{P} = \frac{2}{T^2}\left\{\frac{T_1^2 - T_0^2}{2} - \int_{T_0}^{T} tE\left(\frac{V_0}{m(t)}\right)dt + \int_{T_1}^{T} tE\left(\frac{V_1}{m(t)}\right)dt\right\} \tag{12}$$

The PML build-up m can take a variety of forms. It may increase in the same fashion as v, or it may be essentially constant.[7] In the case of projects with *testing*, m may experience a jump at the start of the testing period (see Section 7). In this case r also changes, so the testing period must treated separately. Rapid changes in $m(t)$ usually indicate a change in the exposure (perils, property or coverage) and a possible change in r.

3. SPECIAL CASES OF $\boldsymbol{m(t)}$

In this section we continue to assume that $v(t) = (V/T)t$ is linear. It is usually difficult to specify the actual PML build-up $m(t)$. However, simple assumptions can often be made. We consider two cases: when m is linear, and when m is constant.

CASE 1: $\boldsymbol{m(t) = (M/T)t}$ ***is linear.***[8] Then (12) becomes:

$$\frac{L}{P} = \frac{2}{T^2}\left\{\frac{T_1^2 - T_0^2}{2} - \int_{T_0}^{T} tE\left(\frac{V_0 T}{Mt}\right)dt + \int_{T_1}^{T} tE\left(\frac{V_1 T}{Mt}\right)dt\right\} \tag{13}$$

The limits of integration T_0 and T_1 are easily determined using the linearity of $m(t)$. Since $V_i = m(T_i) = (M/T)T_i$, we have:

$$\begin{aligned} T_0 &= (T/M)V_0 \\ T_1 &= (T/M)V_1 \end{aligned} \tag{14}$$

Substituting these where they appear in (13) we obtain:

$$\begin{aligned} \frac{L}{P} &= \frac{2}{T^2}\left\{\frac{T^2(V_1^2 - V_0^2)}{2M^2} - \int_{TV_0/M}^{T} tE\left(\frac{V_0 T}{Mt}\right)dt + \int_{TV_1/M}^{T} tE\left(\frac{V_1 T}{Mt}\right)dt\right\} \\ &= \frac{V_1^2 - V_0^2}{M^2} - \frac{2}{T^2}\int_{TV_0/M}^{T} tE\left(\frac{V_0 T}{Mt}\right)dt + \frac{2}{T^2}\int_{TV_1/M}^{T} tE\left(\frac{V_1 T}{Mt}\right)dt \end{aligned}$$

Now make the change of variable $u = Mt/V_0T$ and $u = Mt/V_1T$ in the first and second integrals, respectively, to obtain:

$$\frac{L}{P} = \frac{V_1^2 - V_0^2}{M^2} - \frac{2V_0^2}{M^2}\int_1^{M/V_0} uE\left(\frac{1}{u}\right)du + \frac{2V_1^2}{M^2}\int_1^{M/V_1} uE\left(\frac{1}{u}\right)du$$

[7] A more complicated example is provided by a hydroelectric power plant. At the start of the project $m(t)$ increases slowly during site preparation and infrastructure works, reaches a peak mid-way during the period (corresponding to collapse of a cofferdam or diversion tunnel, resulting in a catastrophic flood), decreases after the main dam is completed and the diversion tunnels are closed, and increases again during the testing period. The risk exposures change significantly during the period; therefore, r is not constant. An accurate layer price requires a number of separate calculations. As m is not non-decreasing, (4) would require modification; however, (5) remains valid.

[8] This may be a reasonable assumption for the construction of buildings, dams, bridges, etc.

$$= \frac{V_1^2}{M^2}\left\{1 + 2\int_1^{M/V_1} uE\left(\frac{1}{u}\right) du\right\} - \frac{V_0^2}{M^2}\left\{1 + 2\int_1^{M/V_0} uE\left(\frac{1}{u}\right) du\right\}$$

Note that T has disappeared. (T does not affect the layer price; we can always make $T = 1$ by appropriate choice of units or the change of variable $t' = t/T$.)

Define for $x \geq 1$:

$$\mathbb{G}(x) = \int_1^x uE\left(\frac{1}{u}\right) du$$

Then we may write:

$$\frac{L}{P} = \frac{V_1^2}{M^2}\left\{1 + 2\mathbb{G}\left(\frac{M}{V_1}\right)\right\} - \frac{V_0^2}{M^2}\left\{1 + 2\mathbb{G}\left(\frac{M}{V_0}\right)\right\} \tag{15}$$

Unfortunately $\mathbb{G}(x)$ cannot be expressed in terms of elementary functions when $E(x)$ is given by (9).

> *Example:* Suppose v and m build up linearly, with $M = 80$. Calculate L/P for a 40 XS 10 layer, using the *Lloyd's curve:*
>
> $$E(x) = (2/11)\ln\{1 + 323.4549(1 - e^{-1.4x})\}$$
>
> *Solution:* From (15) we have:
>
> $$\frac{L}{P} = \frac{25}{64}\left\{1 + 2\int_1^{1.6} uE\left(\frac{1}{u}\right) du\right\} - \frac{1}{64}\left\{1 + 2\int_1^{8} uE\left(\frac{1}{u}\right) du\right\} \approx 0.1873$$
>
> Note that a naive calculation which assumes $m(t) = 80$ (constant) gives $L/P = E(5/8) - E(1/8) \approx 0.2320$, which overestimates the correct amount by ~24%.

Incidentally, $\mathbb{G}(M/V_0) = \int_1^{M/V_0} uE(1/u)du$ does not converge as $V_0 \to 0$ since $uE(1/u) \geq u(1/u) = 1$. However, the product $V_0^2\mathbb{G}(M/V_0)$ *does* converge as $V_0 \to 0$. By L'Hôpital's rule:

$$\lim_{V_0 \to 0} V_0^2 \int_1^{M/V_0} uE\left(\frac{1}{u}\right) du = \lim_{V_0 \to 0} \frac{\frac{M}{V_0} \cdot E\left(\frac{V_0}{M}\right)\left(-\frac{M}{V_0^2}\right)}{-\frac{2}{V_0^3}} = \frac{M^2}{2}\lim_{V_0 \to 0} E\left(\frac{V_0}{M}\right) = 0$$

CASE 2: $\boldsymbol{m(t) = M}$ ***is constant.***[9] Strictly speaking, this is unrealistic if $v(t) = (V/T)t$ is linear. Since $v(0) = 0$, we have $m(t) > v(t)$ near $t = 0$, which is not possible. This objection can be overcome by choosing:

[9] This may be a reasonable assumption for the construction of "long linear" risks such as roads, railways, tunnels, pipelines, transmission and distribution lines, etc.

$$m(t) = \begin{cases} (V/T)t & 0 \le t < MT/V \\ M & MT/V \le t \le T \end{cases}$$

as illustrated in Figure 2 below.

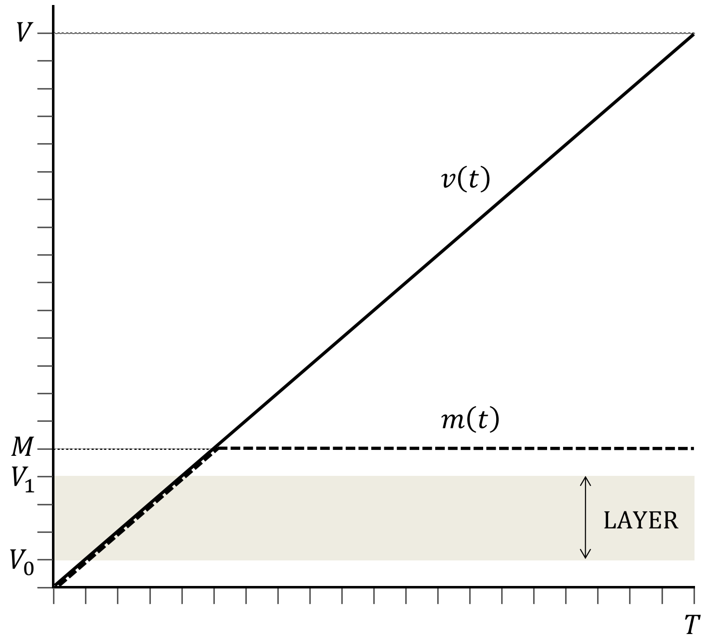


FIGURE 2

In actuality, we can often ignore this complication. The assumption that m is constant usually occurs in situations where m builds up quickly to a constant value $M \ll V$. We can ignore the region near $t = 0$ where $v(t) < M$, since this region has little influence on the layer price (the premium density $rv(t)$ is small). The layer price is dominated by the region where $v(t) > M$.

If m is constant, T_0 and T_1 are not well defined, since $m(t)$ does not enter and exit the layer. We can proceed directly to write:

$$L = \int_0^T \left\{ E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right) \right\} rv(t)dt$$

$$= \left\{ E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right) \right\} \int_0^T rv(t)dt$$

$$= P \left\{ E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right) \right\}$$

which yields the simple formula:

$$\boxed{\frac{L}{P} = E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right) \qquad (16)}$$

This is the usual formula for a static property risk. The behavior of $v(t)$ is irrelevant; what matters is that $m(t)$ is constant, so the factor $E(V_1/m(t)) - E(V_0/m(t)) = E(V_1/M) - E(V_0/M)$ is time-independent.

4. S-SHAPED BUILD-UP OF VALUE (I)

The build-up of value at a construction project typically has a sigmoidal (S-shaped) curve. Costs build up slowly at the start of the project, during mobilization and site preparation. Later on, costs accumulate at an almost constant rate with work crews on site and delivery of construction materials and machinery and equipment to be erected. As the project nears completion, the cost accumulation decelerates.

Recall in Section 2 we used the linearity of v to find $r = 2P/VT$. This actually remains valid if the S-curve is symmetrical under a 180° rotation, as shown in Figure 3.

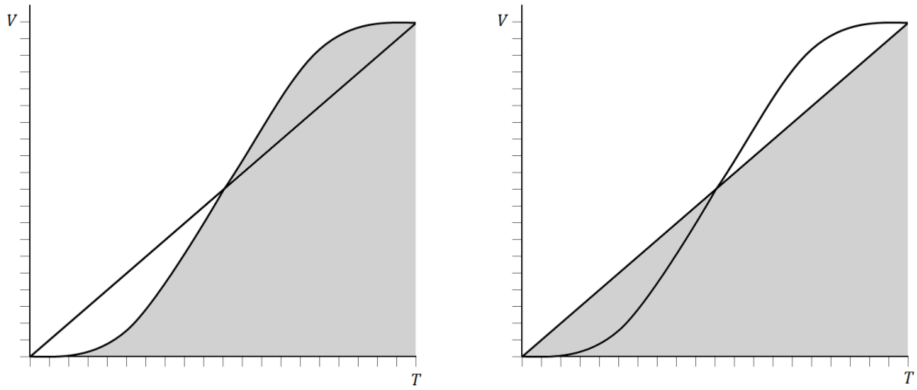


FIGURE 3

To see this, note the area under the S-shaped curve is the same as under the straight line, so $\int_0^T v(t)dt = VT/2$. In fact, this is true for any function $v(t)$ that is symmetrical under a 180° rotation. ***Symmetry under a 180° rotation*** means $v(t) + v(T-t) = V$. Hence:

$$\begin{aligned} 2\int_0^T v(t)dt &= \int_0^T v(t)dt + \int_0^T [V - v(T-t)]dt \\ &= \int_0^T v(t)dt + VT - \int_0^T v(T-t)dt \\ &= VT \end{aligned}$$

so that $\int_0^T v(t)dt = VT/2$. Therefore, when $v(t)$ is symmetrical under a 180° rotation:

$$r = P\left\{\int_0^T v(t)dt\right\}^{-1} = \frac{2P}{VT}$$

as claimed. This proves equation (6).

We also used linearity of v to express the premium density:

$$rv(t) = \frac{2P}{VT} \cdot \frac{V}{T} t = \frac{2P}{T^2} t$$

but this no longer holds when v is not linear. Looking at Figure 4 below, we see that in the region $0 \leq t \leq T/2$ (where the straight line is above the S-curve), the straight-line gives a higher premium density $rv(t)$ than the S-shaped curve. Similarly, in the region $T/2 \leq t \leq T$, the straight-line gives a lower premium density.

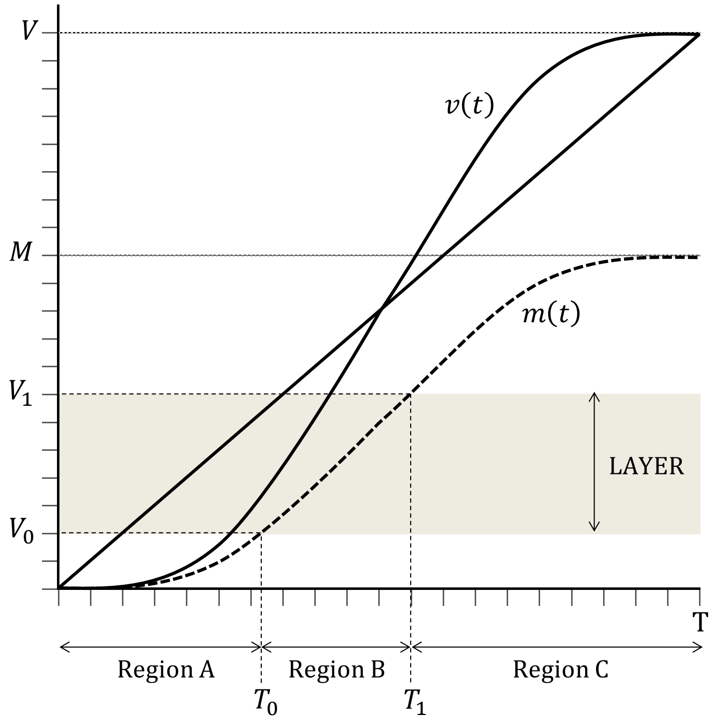


FIGURE 4

These effects do not cancel each other out in the calculation of the layer price, because of the "layer allocation factor" which appears in equations (5) and (6):

$$\theta(t) = E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right)$$

The behavior of $\theta(t)$ as t advances is examined in the Appendix. $\theta(t)$ can favor (i.e., result in a higher layer premium) either the S-shaped curve or the straight-line build-up, depending on the relative widths and positions of regions A, B and C. As the attachment point rises, region A (which contributes nothing) becomes larger, regions B and C shift further to the right, and region C gets smaller; all of which increase the premium for the S-shaped build-up. In most cases, the linear approximation results in a lower premium than an S-shaped function.

5. S-SHAPED BUILD-UP OF VALUE (II)

The S-shaped curve for build-up of value at some construction projects with final value V and period T can be approximated by the functions:

$$u_1(t) = \frac{V}{T^2}\left\{3t^2 - \left(\frac{2}{T}\right)t^3\right\}$$

$$u_2(t) = \frac{V}{2}\left\{\sin\pi\left(\frac{t}{T} - \frac{1}{2}\right) + 1\right\}$$

for $0 \le t \le T$. These functions are very similar on $[0,T]$ as shown in Figure 5 below (with $V = T = 1$). The polynomial u_1 is something of a curiosity; besides possessing 180° symmetry on the interval $[0,T]$ it also approximates u_2 extremely well. The functions u_1 and u_2:

(i) Are symmetric under a 180° rotation

(ii) Satisfy $u_i'(0) = u_i'(T) = 0$

(iii) Have maximum slopes $u_1'(T/2) = 3V/2T$ and $u_2'(T/2) = \pi V/2T$.

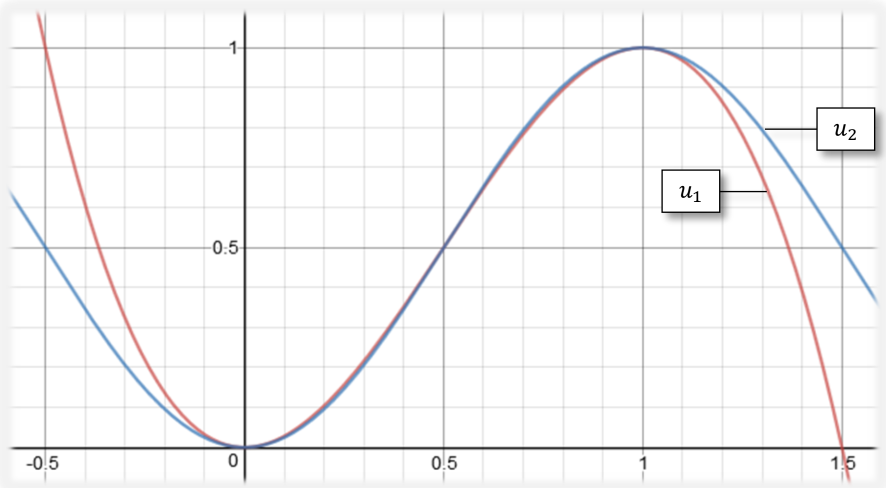


FIGURE 5

Since $r = 2P/VT$ when v is symmetrical under a 180° rotation, we have from (6):

$$\frac{L}{P} = \frac{2}{VT}\int_0^T \left\{E\left(\min\left\{\frac{V_1}{m(t)},1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)},1\right\}\right)\right\} u_i(t)dt \tag{17}$$

Alternatively, we can use the explicit form in (4) and write:

$$\frac{L}{P} = \frac{2}{VT}\left\{\int_{T_0}^{T_1} u_i(t)dt - \int_{T_0}^{T} E\left(\frac{V_0}{m(t)}\right) u_i(t)dt + \int_{T_1}^{T} E\left(\frac{V_1}{m(t)}\right) u_i(t)dt\right\} \tag{18}$$

(Note the factor of V^{-1} in front of the integrals cancels the V in $u_i(t)$.) The limits of integration T_0 and T_1 can be calculated by solving the equations:

$$m(T_0) = V_0$$

$$m(T_1) = V_1$$

We consider again the special cases when m is proportional to v and when m is constant.

CASE 1: Assume m is proportional to v, that is $m(t) = (M/V)u_i(t)$. Then equation (17) or (18) can be used to compute L.

Example: Suppose v and m are given by $v(t) = 100(3t^2 - 2t^3)$ and $m(t) = 80(3t^2 - 2t^3)$ for $0 \le t \le 1$. Calculate L/P for a 40 XS 10 layer, using the Lloyd's curve.

Solution: We solve the equations

$$3T_0^2 - 2T_0^3 = 1/8$$

$$3T_1^2 - 2T_1^3 = 5/8$$

numerically to obtain $T_0 \approx 0.2211$ and $T_1 \approx 0.5841$. Using (18) with $T = 1$:

$$\frac{L}{2P} = \left(t^3 - \frac{t^4}{2}\right)\Big|_{T_0}^{T_1} - \int_{T_0}^{1} E\left(\frac{1/8}{3t^2 - 2t^3}\right)(3t^2 - 2t^3)dt + \int_{T_1}^{1} E\left(\frac{5/8}{3t^2 - 2t^3}\right)(3t^2 - 2t^3)dt$$

$$= 0.1315 - \int_{0.2211}^{1} E\left(\frac{1/8}{3t^2 - 2t^3}\right)(3t^2 - 2t^3)dt + \int_{0.5841}^{1} E\left(\frac{5/8}{3t^2 - 2t^3}\right)(3t^2 - 2t^3)dt$$

Using the Lloyd's curve $E(x) = (2/11)\ln\{1 + 323.4549(1 - e^{-1.4x})\}$ we obtain:

$$\frac{L}{P} \approx 2(0.1315 - 0.3804 + 0.3475) = 0.1971$$

The figure of 0.1873 obtained in the earlier example using a straight line build-up of value is too low by about 5%.

CASE 2: Assume $m(t) = M$ is constant. We note again that $v(t) < M$ near $t = 0$ is unrealistic. If desired, we can choose:

$$m(t) = \begin{cases} v(t) & 0 \le t < \tau \\ M & \tau \le t \le T \end{cases}$$

where $v(\tau) = M$. We ignore this complication by assuming $M \ll V$.

From (16) we know that $v(t)$ does not matter, and:

$$\frac{L}{P} = E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right)$$

This can be checked directly. For example, for u_1:

$$\frac{L}{P} = \frac{1}{P}\int_0^T \left\{E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right)\right\} r u_1(t)dt$$

$$= \frac{1}{P}\int_0^T \left\{E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right)\right\} \cdot \frac{2P}{VT} \cdot \frac{V}{T^2}\left\{3t^2 - \left(\frac{2}{T}\right)t^3\right\} dt$$

$$= \frac{2}{T^3}\left\{E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right)\right\}\int_0^T \left(3t^2 - \frac{2t^3}{T}\right)dt$$

$$= E\left(\frac{V_1}{M}\right) - E\left(\frac{V_0}{M}\right)$$

6. OTHER S-SHAPED CURVES

We have seen that u_1 and u_2 have S-shaped curves on $[0,T]$ which are symmetric under a 180° rotation. Unfortunately, these functions do not admit easy modification of their shape to model different rates of build-up of value.

The build-up of value $v(t)$ for heavy industrial projects like power plants and oil refineries usually has a more pronounced S-shape than for buildings, dams or bridges. At the beginning of such projects, $v(t)$ increases slowly while mobilization and site preparation take place, then increases more rapidly as machinery and equipment are delivered and erected, and flattens out again while numerous functional checks and tests are carried out. A collection of S-shaped curves with different degrees of "steepness" would enable us to model the build-up of value for a wide variety of construction projects.

A family of symmetric S-shaped curves $D_k(t)\colon [0,T] \to [0,V]$ can be generated for $k > 0$ by

$$D_k(t) = \frac{V}{2}\left\{\frac{\tanh\left[k\left(\frac{t}{T} - \frac{1}{2}\right)\right]}{\tanh\left(\frac{k}{2}\right)} + 1\right\} \tag{19}$$

These functions have a maximum slope

$$D_k'(T/2) = \frac{kV}{2T\tanh(k/2)}$$

We define

$$D_0(t) \equiv \lim_{k\to 0} D_k(t) = (V/T)t$$

This can be verified by replacing $\tanh x$ by its Taylor series $\tanh x = x - x^3/3 + \cdots$ which converges for $|x| < \pi/2$:

$$\lim_{k\to 0}\frac{V}{2}\left\{\frac{\tanh\left[k\left(\frac{t}{T} - \frac{1}{2}\right)\right]}{\tanh\left(\frac{k}{2}\right)} + 1\right\} = \lim_{k\to 0}\frac{V}{2}\left\{\frac{k\left(\frac{t}{T} - \frac{1}{2}\right) + \text{terms of order } k^3}{\frac{k}{2} + \text{terms of order } k^3} + 1\right\} = \frac{V}{T}t$$

Several of these curves are illustrated in Figure 6 below (with $V = T = 1$), together with $u_1(t) = 3t^2 - 2t^3$.

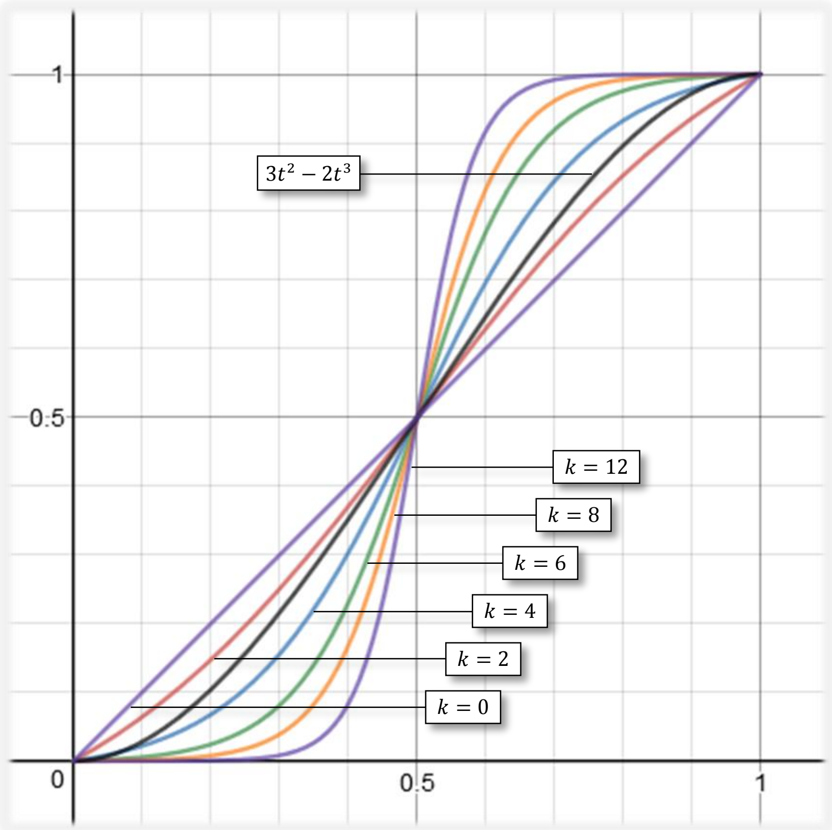


FIGURE 6

Equation (6) naturally applies with the functions D_k. Thus, when the build-up of value $v(t)$ is approximated by $D_k(t)$:

$$\frac{L}{P} = \frac{2}{VT}\int_0^T \left\{E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right)\right\} D_k(t)dt \tag{20}$$

The table below compares the premium allocation for two layers and three different $v(t)$: a straight line, u_1 and D_7. We choose $M = V = 100$ and $m(t) = v(t)$.[10] Note that the linear approximation underestimates the layer price when the build-up has an S-shape, and the error increases as the attachment point increases.

Layer	Layer allocation L/P using $v(t) =$		
	$\frac{Vt}{T}$	$\frac{V}{T^2}\left(3t^2 - \frac{2t^3}{T}\right)$	$\frac{V}{2}\left\{\frac{\tanh[7(t/T - 1/2)]}{\tanh(7/2)} + 1\right\}$
50 XS 10	0.219	0.229	0.244
50 XS 50	0.317	0.417	0.562

[10] The Lloyd's curve is used for $E(x)$.

7. EXPOSURES AND PERILS

Testing periods. Construction risks involving the erection of machinery and equipment usually have a *testing period* that commences after the construction works are complete. The build-up curve, therefore, reaches its maximum value prior to the end of the period and remains constant, as shown in Figure 7 below. The testing period may be responsible for a significant proportion (even a majority) of the policy premium; therefore, it plays an important role in layer pricing.

The construction and testing periods must be priced separately, because r is not the same during these two phases. Hence, separate premium calculations must be carried out for the construction period (with its build-up curve $v_{\text{con}}(t)$ and associated premium P_{con}) and the testing period (with $v_{\text{test}}(t) = V$ and associated premium P_{test}).

Note that the PML curve m usually jumps to a higher value at the start of the testing period, as typically $M_{\text{test}} > M_{\text{con}}$ due to the presence of more severe loss exposures during testing.

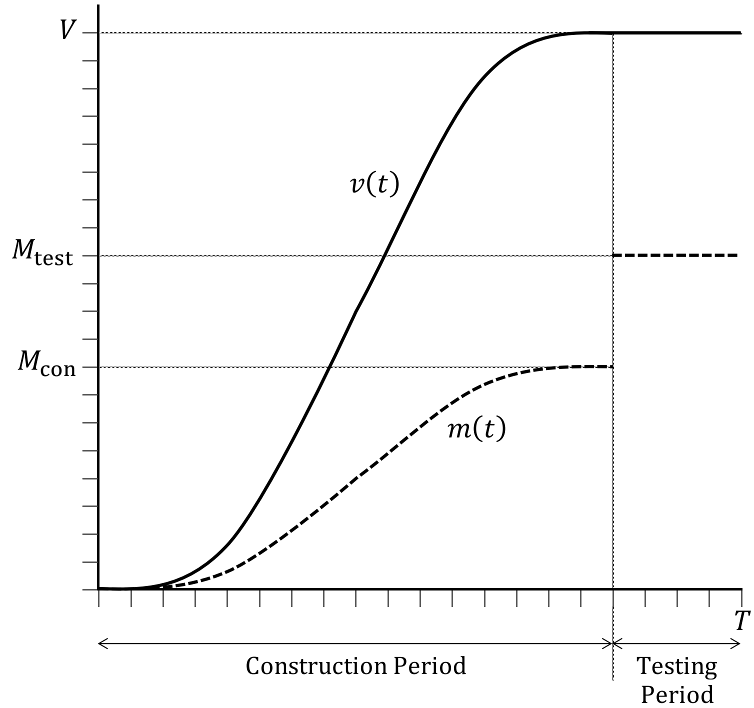


FIGURE 7

Natural catastrophe exposures. Natural catastrophe exposures must be treated separately if they contribute significantly to the premium, because they have their own exposure rating curves, PML curves and r values.

Some catastrophe perils (e.g., earthquake) have a PML curve that can be modeled as a fixed proportion of the value: $m_{\text{cat}}(t) = \alpha v(t)$. Other catastrophe perils, such as hurricane, may be seasonal and modeled with a PML $m_{\text{cat}}(t) = \alpha v(t)$ during hurricane seasons and $m_{\text{cat}}(t) = 0$ between seasons, as shown in Figure 8 below.

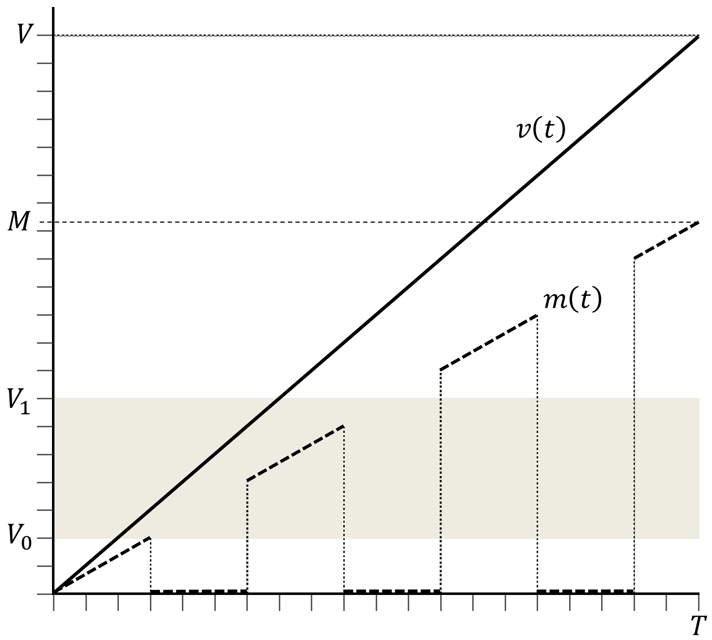


FIGURE 8

Note that here r is not constant: $r(t) = \tilde{r}\chi(t)$ for some constant $\tilde{r}$, where

$$\chi(t) = \begin{cases} 1 & t \in \text{hurricane season} \\ 0 & t \notin \text{hurricane season} \end{cases}$$

Hence:

$$P = \int_0^T r(t)v(t)dt = \int_0^T \tilde{r}\chi(t)v(t)dt$$

and therefore:

$$\tilde{r} = P\left\{\int_0^T \chi(t)v(t)dt\right\}^{-1} \tag{21}$$

Unfortunately, $\chi(t)v(t)$ is not symmetrical under a 180° rotation, so we cannot evaluate this expression as we did in the symmetrical case where $r = 2P/VT$. Instead $\tilde{r}$ must be calculated using (21). From (5) we have:

$$L = \int_0^T \left\{E\left(\min\left\{\frac{V_1}{m_{\text{cat}}(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m_{\text{cat}}(t)}, 1\right\}\right)\right\}\tilde{r}\chi(t)v(t)dt$$

Note that outside the hurricane season $m_{\text{cat}}(t) = 0$, so the term in curly brackets is $1 - 1 = 0$. Hence, the integral is *over hurricane seasons only*, so the factor $\chi(t)$ is superfluous. We can therefore simplify this formula to:

$$L = \int_0^T \left\{E\left(\min\left\{\frac{V_1}{m_{\text{cat}}(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m_{\text{cat}}(t)}, 1\right\}\right)\right\}\tilde{r}v(t)dt \tag{22}$$

Note that the PML function $m_{\text{cat}}(t)$ has the form $m_{\text{cat}}(t) = \chi(t)m(t)$ for some function $m(t)$.

Independent perils and exposures. Suppose the overall loss exposure is comprised of multiple, independent[11] exposures $\mathcal{E}_i$ with associated $m_i(t)$, $E_i(x)$, r_i and P_i. (The exposure rating curve $E_i(x)$ is parameterized by some c_i). Then the layer price is calculated by adding the layer prices for each individual exposure:

$$L = \sum_i L_i = \sum_i \int_0^T \left\{E_i\left(\min\left\{\frac{V_1}{m_i(t)}, 1\right\}\right) - E_i\left(\min\left\{\frac{V_0}{m_i(t)}, 1\right\}\right)\right\} r_i v(t)dt$$

If $v(t)$ is symmetrical under a 180° rotation:

$$L = \frac{2}{VT}\sum_i P_i \int_0^T \left\{E_i\left(\min\left\{\frac{V_1}{m_i(t)}, 1\right\}\right) - E_i\left(\min\left\{\frac{V_0}{m_i(t)}, 1\right\}\right)\right\} v(t)dt \qquad (23)$$

8. Loss limits

An insurance policy may have *loss limits* (particularly in the case of natural catastrophe perils) that limit the maximum payout for an event. Suppose there is a loss limit $Q < M$. (If $Q \geq M$, the limit serves no purpose.) We may assume $V_1 \leq Q$. (If $V_1 > Q$, we can simply redefine $V_1 = Q$.)

The layer premium L is ***not affected by the loss limit***, since the layer lies below the limit ($V_1 \leq Q$). However, the premium $\tilde{P}$ *with* the loss limit is not the same as the premium P *without* the loss limit (it is smaller). The underwriter can supply the actual policy premium $\tilde{P}$, but probably not P. So the general formula (5) needs to be altered so that $\tilde{P}$ appears instead of P.

Equation (6) for a primary layer gives:

$$\tilde{P} = \int_0^T E\left(\min\left\{\frac{Q}{m(t)}, 1\right\}\right) r v(t)dt \qquad (24)$$

so:

$$r = \tilde{P}\left\{\int_0^T E\left(\min\left\{\frac{Q}{m(t)}, 1\right\}\right) v(t)dt\right\}^{-1}$$

Therefore, by (5):

[11] "Independent" means that a loss cannot arise from more than one peril or exposure. The various phases of the project (contract works, testing, maintenance) are independent because they do not overlap. Similarly, natural catastrophe perils (earthquake, windstorm, flood) are independent of one another and independent of any other peril or exposure.

$$L = \frac{\tilde{P}\int_0^T \left\{E\left(\min\left\{\frac{V_1}{m(t)},1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)},1\right\}\right)\right\} v(t)dt}{\int_0^T E\left(\min\left\{\frac{Q}{m(t)},1\right\}\right) v(t)dt} \tag{25}$$

9. Delay in Start-Up

Delay in Start-Up (DSU) can be treated similarly to property damage (PD) coverage.

As time advances during the project period, DSU claims are generally ***more likely to occur and be more severe***. This is because of:

- Increasing value on site
- Increasing scope and complexity of the project works
- Emergence of a well-defined critical path in the schedule with interdependences and "bottlenecks"
- Erosion or exhaustion of any buffer/margin built into the original schedule, due to inevitable, routine delays
- Decreasing time remaining during which to make up time and reduce a delay

This implies ***the premium density for DSU should increase with time.*** DSU coverage has its own sum insured $\hat{V}$, but no "build-up of value" analogous to $v(t)$. (A hat ^ will denote quantities applying to DSU.) The full DSU sum insured is theoretically exposed at any time. However, we do not choose $\hat{v}(t) = \hat{V}$ (constant) for $0 \le t \le T$, since the resulting premium density $\hat{r}\hat{v}(t)$ would overweight premium in the early part in the period.

We make the simple (but not unreasonable assumption) that the premium density increases in proportion to the project value: $\hat{r}\hat{v}(t) \propto v(t)$. This requires:

$$\hat{v}(t) = \frac{\hat{V}}{V} \cdot v(t) \tag{26}$$

With this choice, note that:

$$\hat{r}\hat{v}(t) = \frac{\hat{P}\hat{v}(t)}{\int_0^T \hat{v}(t)\,dt} = \frac{\hat{P}v(t)}{\int_0^T v(t)\,dt} = \frac{\hat{P}rv(t)}{\int_0^T rv(t)\,dt} = \frac{\hat{P}}{P} \cdot rv(t) \tag{27}$$

which can be written symmetrically:

$$\frac{\hat{r}\hat{v}(t)}{\hat{P}} = \frac{rv(t)}{P}$$

The layer price $\hat{L}$ for ***standalone*** DSU coverage follows immediately from (5), using (26) and (27):

$$\hat{L} = \int_0^T \left\{\hat{E}\left(\min\left\{\frac{V_1}{\hat{m}(t)}, 1\right\}\right) - \hat{E}\left(\min\left\{\frac{V_0}{\hat{m}(t)}, 1\right\}\right)\right\} \hat{r}\hat{v}(t)dt$$

$$= \frac{\hat{P}r}{P} \int_0^T \left\{\hat{E}\left(\min\left\{\frac{V_1}{\hat{m}(t)}, 1\right\}\right) - \hat{E}\left(\min\left\{\frac{V_0}{\hat{m}(t)}, 1\right\}\right)\right\} v(t)dt$$

If $v(t)$ is symmetrical under a 180° rotation, then $r = 2P/VT$ so:

$$\boxed{\hat{L} = \frac{2\hat{P}}{VT} \int_0^T \left\{\hat{E}\left(\min\left\{\frac{V_1}{\hat{m}(t)}, 1\right\}\right) - \hat{E}\left(\min\left\{\frac{V_0}{\hat{m}(t)}, 1\right\}\right)\right\} v(t)dt \qquad (28)}$$

We write $\hat{E}$ since the exposure rating curve for DSU is different than for PD. (Generally $\hat{c} < c$ since DSU has a higher proportion of large losses compared to PD.)

An accurate PML[12] build-up curve is even more challenging to specify for DSU than for property damage, because DSU PML scenarios are difficult to determine and model. In the discussion which follows we leave $\hat{m}(t)$ arbitrary. However, we expect $\hat{m}(t)$ to be increasing, since the severity of DSU claims increases with time. One option is to choose $\hat{m}$ linear:

$$\hat{m}(t) = \frac{\hat{M}}{T} t$$

where $\hat{M}$ is the maximum PML (often, but not always, equal to $\hat{V}$). Another option is to choose $\hat{m}$ proportional to m or v:[13]

$$\hat{m}(t) = \frac{\hat{M}}{M} \cdot m(t) \quad \text{or} \quad \hat{m}(t) = \frac{\hat{M}}{V} \cdot v(t) = \frac{\hat{M}}{\hat{V}} \cdot \hat{v}(t)$$

Any of these assumptions may be used in (28).[14]

10. Combined PD and DSU coverage

As explained at the end of Section 7, when the overall loss exposure is comprised of independent exposures $\mathcal{E}_i$, the layer premium is calculated by adding the layer premiums for each individual exposure:

$$L = \sum_i L_i$$

The DSU layer premium in (28), however, ***cannot be added to the PD layer premium*** to arrive at the premium for combined PD and DSU coverage. This is because a DSU claim necessarily occurs together with the PD claim which triggered the delay (they are ***dependent*** events). Both claims contribute to erosion of the attachment point and the layer.

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[12] In the case of Delay in Start-Up, the PML is sometimes called the *Maximum Probable Delay* (MPD).

[13] These produce the same function $\hat{m}$ if $m \propto v$.

[14] If $m(t)$ is constant (e.g., for roads, railways, tunnels, etc.) then $\hat{m}(t) = (\hat{M}/M)m(t)$ would not be appropriate, because $\hat{m}(t)$ should still be increasing.

Therefore, when pricing non-proportional insurance with PD and DSU coverage, we must consider that the layer is exposed to PD and DSU losses combined, not separately. Unfortunately, no simple combination of the layer prices calculated in (5) and (28) can provide the correct price for combined PD and DSU coverage, since the time-evolution of the values and PMLs must be considered.

We approach the problem by adding the PD and DSU exposures to create a ***combined exposure*** with Probable Maximum Loss $m_+(t) = m(t) + \widehat{m}(t)$. We also write $M_+ = M + \widehat{M}$.

The combined exposure has a loss distribution described by an exposure rating curve $E_+(x) = E_{c_+}(x)$, which lies in-between the curves $E(x) = E_c(x)$ and $\hat{E}(x) = E_{\hat{c}}(x)$. A simple way to select c_+ is to create a weighted average of c and $\hat{c}$:[15]

$$c_+ = c\left(\frac{M}{M + \widehat{M}}\right) + \hat{c}\left(\frac{\widehat{M}}{M + \widehat{M}}\right) = \frac{cM + \hat{c}\widehat{M}}{M_+}$$

During the time interval $[t, t + dt]$ the layer attracts a portion of the PD premium $rv(t)dt$ and the DSU premium $\hat{r}\hat{v}(t)dt$. Hence:

$$L_+ = \int_0^T \left\{E_+\left(\min\left\{\frac{V_1}{m_+(t)}, 1\right\}\right) - E_+\left(\min\left\{\frac{V_0}{m_+(t)}, 1\right\}\right)\right\}\big(rv(t) + \hat{r}\hat{v}(t)\big)dt \tag{29}$$

Inserting (27) into (29) we obtain:

$$\boxed{L_+ = \frac{P_+ r}{P}\int_0^T \left\{E_+\left(\min\left\{\frac{V_1}{m_+(t)}, 1\right\}\right) - E_+\left(\min\left\{\frac{V_0}{m_+(t)}, 1\right\}\right)\right\}v(t)dt} \tag{30}$$

where $P_+ \equiv P + \hat{P}$. Inserting (1) for r gives an equivalent, more symmetrical formula:[16]

$$L_+ = \frac{P_+}{\int_0^T v(t)dt}\int_0^T \left\{E_+\left(\min\left\{\frac{V_1}{m_+(t)}, 1\right\}\right) - E_+\left(\min\left\{\frac{V_0}{m_+(t)}, 1\right\}\right)\right\}v(t)dt \tag{31}$$

---

[15] Theoretically we could define $c_+ = c_+(t)$ *at each time t:*

$$c_+(t) = c\left(\frac{m(t)}{m(t) + \widehat{m}(t)}\right) + \hat{c}\left(\frac{\widehat{m}(t)}{m(t) + \widehat{m}(t)}\right) = \frac{cm(t) + \hat{c}\widehat{m}(t)}{m_+(t)}$$

[16] Replacing $v(t)$ with any multiple $\alpha v(t)$ leaves (30) unchanged, because the constants cancel in the two integrals. Therefore, we can replace $v(t)$ with $(1 + \hat{V}/V)v(t) = v(t) + \hat{v}(t) \equiv v_+(t)$ to obtain the perfectly symmetrical formula:

$$L_+ = \frac{P_+}{\int_0^T v_+(t)dt}\int_0^T \left\{E_+\left(\min\left\{\frac{V_1}{m_+(t)}, 1\right\}\right) - E_+\left(\min\left\{\frac{V_0}{m_+(t)}, 1\right\}\right)\right\}v_+(t)dt$$

If $v(t)$ is symmetrical under a 180° rotation, then (30) or (31) yields:

$$L_+ = \frac{2P_+}{VT} \int_0^T \left\{ E_+ \left(\min\left\{ \frac{V_1}{m_+(t)}, 1 \right\} \right) - E_+ \left(\min\left\{ \frac{V_0}{m_+(t)}, 1 \right\} \right) \right\} v(t) dt \tag{32}$$

Note that (32) reduces to (5) in the case of standalone PD coverage ($P_+ = P$, $m_+ = m$, $E_+ = E$) and to (28) in the case of standalone DSU coverage ($P_+ = \hat{P}$, $m_+ = \widehat{m}$, $E_+ = \hat{E}$).

Multiple independent exposures. Suppose $\mathcal{E}_i$ are independent exposures with associated $m_i(t)$, $E_i(x)$, r_i and P_i (for PD) and $\widehat{m}_i(t)$, $\hat{E}_i(x)$ and $\hat{P}_i$ (for DSU). The combined PD+DSU layer premium is calculated by adding the layer premiums for each PD+DSU exposure using (30):[17]

$$L_+ = \sum_i L_{i+} = \sum_i \frac{P_{i+} r_i}{P_i} \int_0^T \left\{ E_{i+} \left(\min\left\{ \frac{V_1}{m_{i+}(t)}, 1 \right\} \right) - E_{i+} \left(\min\left\{ \frac{V_0}{m_{i+}(t)}, 1 \right\} \right) \right\} v(t) dt$$

If $v(t)$ is symmetrical under a 180° rotation:

$$L_+ = \frac{2}{VT} \sum_i P_{i+} \int_0^T \left\{ E_{i+} \left(\min\left\{ \frac{V_1}{m_{i+}(t)}, 1 \right\} \right) - E_{i+} \left(\min\left\{ \frac{V_0}{m_{i+}(t)}, 1 \right\} \right) \right\} v(t) dt$$

Loss Limits with combined PD and DSU coverage. Suppose an exposure $\mathcal{E}_i$ has a loss limit Q that applies to the combined PD+DSU coverage. As noted in Section 8, we may assume without loss of generality that $V_1 \leq Q < M_+$.

The layer premium L_{i+} is ***not affected by the loss limit***, since the layer lies below the limit ($V_1 \leq Q$). But we need to rewrite (30) so that the *actual* premium $\tilde{P}_{i+} = \tilde{P}_i + \tilde{\hat{P}}_i$ (with the loss limit)[18] appears instead of $P_{i+} = P_i + \hat{P}_i$. Repeating the method of Section 8, we use (30) to obtain the policy premium $\tilde{P}_{i+}$ for exposure $\mathcal{E}_i$ with loss limit Q by choosing a primary layer defined by $V_0 = 0$ and $V_1 = Q$:

$$\tilde{P}_{+i} = \frac{P_{i+} r_i}{P_i} \int_0^T \left\{ E_{i+} \left(\min\left\{ \frac{Q}{m_{i+}(t)}, 1 \right\} \right) \right\} v(t) dt$$

Therefore:

$$\frac{P_{i+} r_i}{P_i} = \tilde{P}_{i+} \left\{ \int_0^T \left\{ E_{i+} \left(\min\left\{ \frac{Q}{m_{i+}(t)}, 1 \right\} \right) \right\} v(t) dt \right\}^{-1}$$

Inserting this back into (30), we obtain:

---

[17] The underwriter should be able to supply P_i, but perhaps not $\hat{P}_i$ (which is needed to specify $P_{i+} = P_i + \hat{P}_i$). It is not uncommon for DSU to be priced as a multiple of the PD price or in some other coarse fashion. If $\hat{P}_i$ is not provided, we can make the simple assumption that $\hat{P}_i = (P_i/P)\hat{P}$. The policy DSU premium $\hat{P}$ should be known.

[18] As above, if $\tilde{\hat{P}}_i$ is not provided, we can assume $\tilde{\hat{P}}_i = (\tilde{P}_i/\tilde{P})\tilde{\hat{P}}$. The policy DSU premium $\tilde{\hat{P}}$ should be known.

$$L_{i+} = \frac{\widetilde{P}_{i+} \int_0^T \left\{ E_{i+}\left(\min\left\{\frac{V_1}{m_{i+}(t)}, 1\right\}\right) - E_{i+}\left(\min\left\{\frac{V_0}{m_{i+}(t)}, 1\right\}\right)\right\} v(t)dt}{\int_0^T \left\{ E_{i+}\left(\min\left\{\frac{Q}{m_{i+}(t)}, 1\right\}\right)\right\} v(t)dt} \tag{33}$$

APPENDIX: THE "ALLOCATION FACTOR" $\boldsymbol{\theta(t)}$

In Section 4 we began a discussion of the "layer allocation factor":

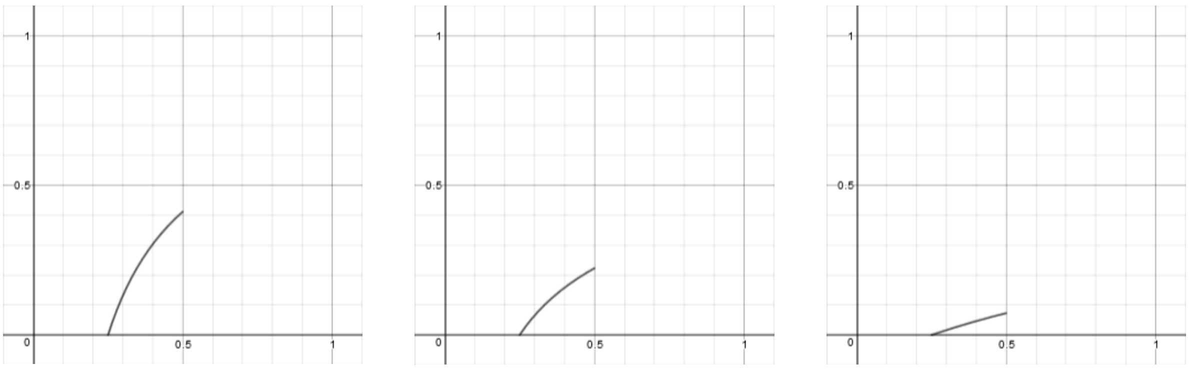
$$\theta(t) = E\left(\min\left\{\frac{V_1}{m(t)}, 1\right\}\right) - E\left(\min\left\{\frac{V_0}{m(t)}, 1\right\}\right)$$

$$= \begin{cases} 0 & m(t) < V_0 & \text{(Region A)} \\ 1 - E(V_0/m(t)) & V_0 \le m(t) < V_1 & \text{(Region B)} \\ E(V_1/m(t)) - E(V_0/m(t)) & V_1 \le m(t) & \text{(Region C)} \end{cases}$$

Now we examine $\theta(t)$ more carefully as t advances. For the sake of simplicity, we assume $T = 1$, and that $m(t) = Mt$ is linear.

Region A. In region A, $\theta(t) = 0$ so there is no contribution.

Region B. In region B, assuming $V_0 > 0$, $\theta(t)$ increases from 0 to $1 - E(V_0/V_1)$ as Mt increases from V_0 to V_1. Graphs of $\theta(t)$ for $V_0/M = 0.25$ and $V_1/M = 0.5$ are shown below for various values of c.

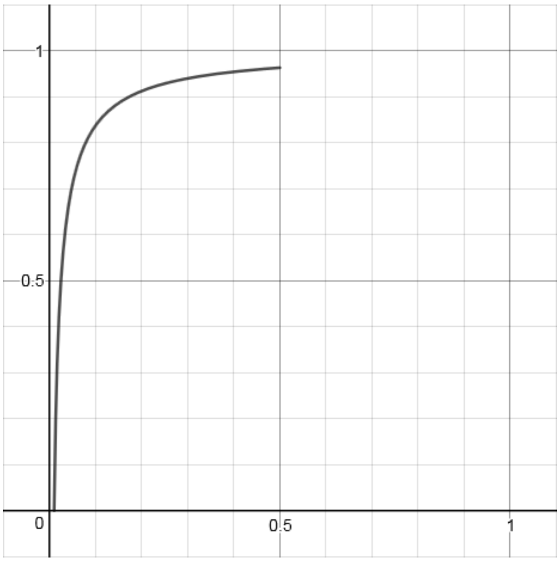


$c = 1$ $\qquad$ $c = 3$ $\qquad$ $c = 5$

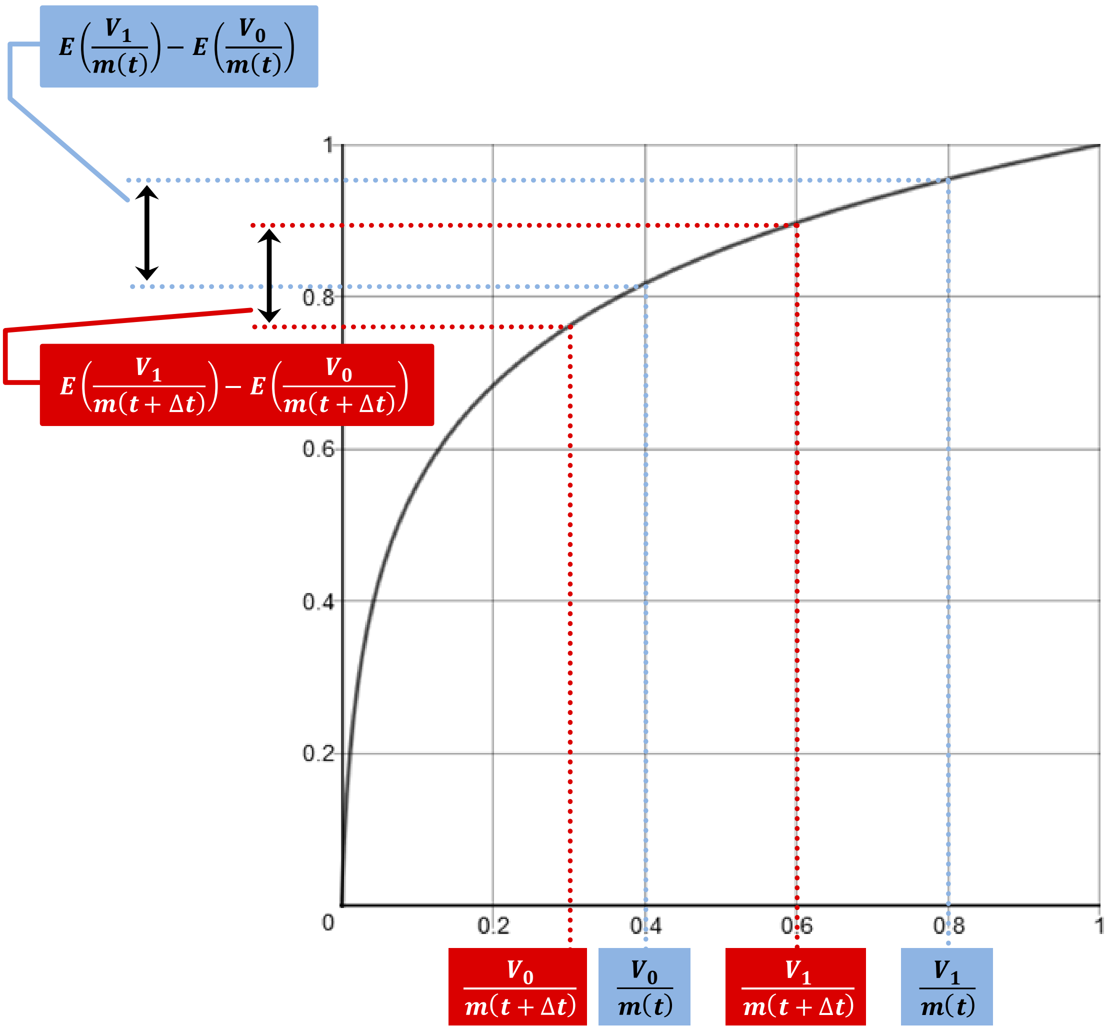
If $V_0 = 0$ (a pure primary layer) then $\theta(t) = 1$. However, for $t > 0$,

$$\lim_{V_0 \downarrow 0} \theta(t) = 1$$

Therefore, $\theta(t)$ — viewed for fixed t as a function of V_0 — is right-continuous at $V_0 = 0$ for all $0 < t \le 1$; it fails to be right-continuous only when $t = 0$. This anomaly can therefore be ignored. The following graph shows $\theta(t)$ for $V_0/M = 0.01$, $V_1/M = 0.5$ and $c = 1$.

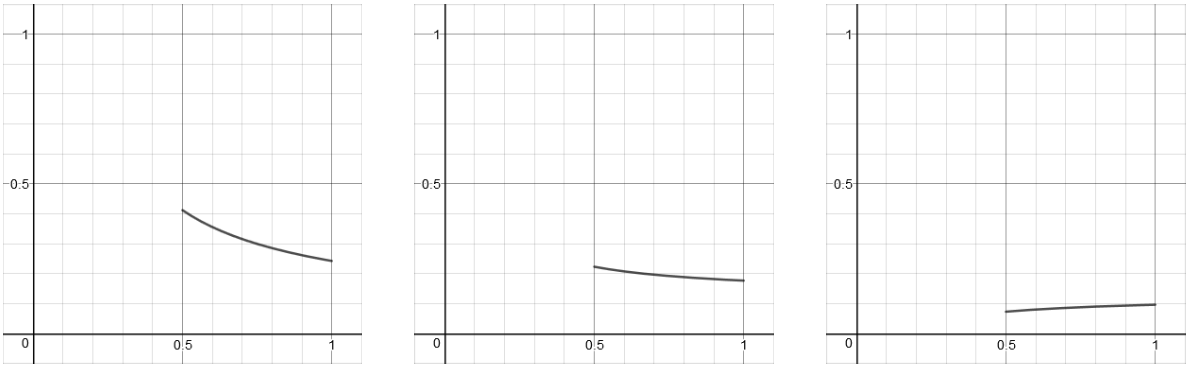


Region C. As t increases, both $E(V_1/Mt)$ and $E(V_0/Mt)$ are decreasing, so $\theta(t)$ is the difference between two decreasing numbers, as illustrated below (with $V_1 = 2V_0$).



The behavior of $\theta(t)$ is not obvious, even for simple $m(t)$ such as a straight line, and turns out to be surprisingly complex. $\theta(t)$ is rather sensitive to the values of V_0/M and V_1/M as well as the parameter c (remember that $E(x) = E_c(x)$).

$\theta(t)$ turns out to be decreasing for $0 \le c < \left(\sqrt{753} - 3\right)/6 = 4.073...$ and increasing for $c > 4.073...$[19] Graphs of $\theta(t)$ for $V_0/M = 0.25$ and $V_1/M = 0.5$ are shown below for various values of c.



$c = 1$ $c = 3$ $c = 5$

Note there is nothing anomalous about the case $V_0 = 0$ in region C: if $V_0 = 0$, $\theta(t)$ decreases steadily from 1 to $E(V_1/M)$ as Mt increases from V_1 to M.

Examining the behavior of $\theta(t)$ in various situations, we conclude that $\theta(t)$ may be increasing or decreasing in region C, but in most cases does not vary too greatly.

[19] This occurs because $\beta(c) = e^{3.1-0.15(1+c)c} = 1$ in equation (8) when $c = \left(\sqrt{753} - 3\right)/6$.